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HYDRAULIC JUMP IN SHEAR FLOW OF A BAROTROPIC FLUID

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We consider a mathematical model in a long-wave approximation that describes the motion of an ideal barotropic fluid layer with a free-boundary. Hydraulic-jump models for both irrotational and rotational flows are formulated. The properties of hydraulic jumps are analyzed. It is shown that, in the general case, there are not only jumps with an increasing downstream fluid level, but also jumps decreasing the flow level.

Hydraulic jumps on a shear incompressible flow were examined in [1]. For the system of equations governing the propagation of long waves in the layer of a barotropic fluid some exact solutions were obtained in [2, 3].

1. Mathematical Model. Let us consider the initial boundary-value problem

$$\begin{gathered} u_T + uu_X + vu_Y + \rho^{-1}p_X = 0, \\ \delta^2(v_T + uv_X + vv_Y) + \rho^{-1}p_Y = -1, \quad 0 \leqslant Y \leqslant h(X,T), \\ \rho_T + u\rho_X + v\rho_Y + \rho(u_X + v_Y) = 0, \quad -\infty < X < \infty, \\ \rho = R(p) \quad (R'(p) > 0), \quad \rho(X,Y,0) = \rho_0(X,Y), \quad u(X,Y,0) = u_0(X,Y), \\ v(X,Y,0) = v_0(X,Y), \quad h(X,0) = h_0(X), \quad v(X,0,T) = 0, \\ h_T + u(X,h,T)h_X = v(X,h,T), \quad p(X,h,T) = p_0 = \text{const}, \end{gathered} \tag{1.1}$$

which describes the plane-parallel motion of an ideal barotropic fluid layer with a free boundary $Y = h(X,T)$ over an even bottom in a gravity field. Here

$$\begin{gathered} u_1 = (gH_0)^{1/2}u, \quad v_1 = (gH_0)^{1/2}H_0L_0^{-1}v, \quad p_1 = R_0gH_0p, \\ \rho_1 = R_0\rho, \quad X_1 = L_0X, \quad Y_1 = H_0Y, \quad T_1 = L_0(gH_0)^{-1/2}T \end{gathered}$$

are the dimensional components of the velocity vector, the pressure, density, the Cartesian coordinates in a plane, and time, respectively; u, v, p, ρ, X, Y, and T are the corresponding dimensionless quantities; the parameters H_0 and L_0 determine the characteristic vertical and horizontal scales; the parameter R_0 has the dimension of density; g is the acceleration of gravity; and $\delta = H_0L_0^{-1}$.

The following problem results from (1.1) as a long-wave approximation ($\delta = 0$), as is shown in [4]:

$$\begin{gathered} u_T + uu_X + vu_Y + h_X = 0, \\ (f(h))_T + \Big(\int_0^h f'(h-Y)\,u(X,Y,T)\,dY\Big)_X = 0, \\ u(X,Y,0) = u_0(X,Y), \quad h(X,0) = h_0(X). \end{gathered} \tag{1.2}$$

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The function f is defined by the equality

$$f\Big(\int\limits_{p_0}^{p}((R(\xi))^{-1})\,d\xi\Big)=p,$$

and the pressure, the density, and the vertical velocity component are given by

$$p=f(h-Y),\quad \rho=f'(h-Y),\quad v=-\rho^{-1}\int\limits_0^Y(\rho_T+(u\rho)_X)\,dY. \tag{1.3}$$

In the theory of long waves, the vanishing of vorticity is equivalent to the condition $u_Y=0$. For irrotational motions, system (1.2) reduces to the analog of one-dimensional gas dynamics

$$\eta(u_T+uu_X)+P_X=0,\quad \eta_T+(u\eta)_X=0, \tag{1.4}$$

where $\eta=f(h)-p_0$ is the pressure variation at the cross section $x=\text{const}$. The dependence

$$P=P(\eta)=\int\limits_0^\eta \xi\,(R(\xi+p_0))^{-1}\,d\xi \tag{1.5}$$

gives the equation of state [4].

In the general case ($u_Y\not\equiv 0$), introduction of mixed Eulerian–Lagrangian independent variables x, λ, and t [4]

$$X=x,\quad T=t,\quad Y=\Phi(x,\lambda,t)$$

transforms system (1.2) to the form

$$u_t+uu_x+\Big(R\Big(p_0+\int\limits_0^1 H\,d\lambda\Big)\Big)^{-1}\int\limits_0^1 H_x\,d\nu=0,\quad H_t+(uH)_x=0\quad(0\leqslant\lambda\leqslant 1,\ -\infty<x<\infty) \tag{1.6}$$

($H=\rho\Phi_\lambda$ is a new unknown function). When the solution of system (1.6) is known, one can find p, ρ, Φ, and v from the relations

$$p=p_0+\int\limits_\lambda^1 H\,d\nu,\quad \rho=R(p),\quad \Phi=\int\limits_0^\lambda \rho^{-1}H\,d\nu,\quad v=\Phi_t+u\Phi_x.$$

At $t=0$ one can set $\Phi=\lambda h_0(x)$, $u=u_0(x,\lambda h_0(x))$, and $\rho=f'((1-\lambda)h_0(x))$. For u and H satisfying special conditions [4], system (1.6) can be treated as a hyperbolic system. Its characteristics are given by the equations

$$\frac{dx}{dt}=k^i\quad(i=1,2),\qquad \frac{dx}{dt}=u(x,\lambda,t)\quad(\lambda=\text{const}).$$

The characteristic roots of the discrete spectrum are determined by the characteristic equation

$$R\Big(p_0+\int\limits_0^1 H\,d\nu\Big)=\int\limits_0^1 H(u-k)^{-2}\,d\nu. \tag{1.7}$$

In the case of irrotational flow ($u_\lambda\equiv 0$), Eq. (1.7) reduces to the characteristic equation of system (1.4): $(u-k)^2=\eta(R(\eta+p_0))^{-1}=P'(\eta)$.

To define the discontinuous solutions of systems (1.4) and (1.6), it is natural to consider the following conservation laws for the of mass and momentum of the fluid layer [the divergent form of Eqs. (1.4)]:

$$(\eta u)_t+(\eta u^2)_x+P_x=0,\quad \eta_t+(u\eta)_x=0.$$

At the shock front $x=x(t)$, the relations below should hold:

$$[\eta(u-D)^2+P]=0,\quad [\eta(u-D)]=0\quad(D=x'(t)), \tag{1.8}$$

where $[f] = f^- - f^+$ is the jump of the function. The internal energy $\varepsilon(\eta)$ is determined by integration of the equatio.. $\varepsilon'(\eta) = \eta^{-2}P$:

$$\varepsilon(\eta) = -\eta^{-1}P + i(\eta), \quad i(\eta) = \int_0^\eta (R(\xi + p_0))^{-1} d\xi. \tag{1.9}$$

From Eqs. (1.4) follows the equation of energy balance

$$(\eta(\varepsilon + 2^{-1}u^2))_t + (u\eta(i + 2^{-1}u^2))_x = 0.$$

At the discontinuity front $x = x(t)$ we require satisfaction of the energy loss condition:

$$\eta(u - D)[2^{-1}(u - D)^2 + i] \geqslant 0. \tag{1.10}$$

The mathematical model of a hydraulic jump for the case of irrotational motion is given by (1.8) and (1.10).

In the general case, the model of a hydraulic jump is constructed by analogy with the model for an incompressible fluid considered in [1]. From Eqs. (1.6) one can obtain the conservation laws

$$H_t + (uH)_x = 0, \quad (H\omega\rho^{-1})_t + (uH\omega\rho^{-1})_x = 0, \quad \Big(\int_0^1 Hu d\nu\Big)_t + \Big(\int_0^1 Hu^2 d\nu\Big)_x + P_x = 0. \tag{1.11}$$

Here $\omega = u_Y = u_\lambda(\Phi_\lambda)^{-1}$ is the vorticity, and the function $P = P(\eta)$ is defined by (1.5), where

$$\eta = \int_0^1 H d\nu. \tag{1.12}$$

The first two Eqs. (1.11) are the local laws of conservation of mass and the quantities ω/ρ [ω/ρ are conserved in particles if we consider the exact model (1.1)]. The last equation expresses the conservation law for the horizontal momentum of the fluid layer. From (1.11) follow the shock relations at the discontinuity front:

$$[H(u - D)] = 0, \quad [H(u - D)\omega\rho^{-1}] = 0, \quad \Big[\int_0^1 H(u - D)^2 d\nu + P\Big] = 0. \tag{1.13}$$

From (1.6) one can also obtain equation of energy balance for the fluid layer:

$$\Big(\int_0^1 H\Big(\varepsilon + \frac{1}{2}u^2\Big) d\nu\Big)_t + \Big(\int_0^1 Hu\Big(i + \frac{1}{2}u^2\Big) d\nu\Big)_x = 0,$$

where $\varepsilon = \varepsilon(\eta)$ and $i = i(\eta)$ are defined by (1.9); and η, by (1.12). In addition, we assume the loss of the layer energy at the front:

$$\int_0^1 H(u - D)[\frac{1}{2}(u - D)^2 + i] d\nu \geqslant 0. \tag{1.14}$$

The model of a hydraulic jump for rotational flow is given by (1.13) and (1.14). It is obvious that conditions (1.13) and (1.14) become (1.8) and (1.10) when $\omega = 0$.

2. Analysis of Shock Relations. System (1.4) has been studied in many papers [5]. It is well known that important features of discontinuous solutions depend on the properties of the function $g(\tau) = P(\tau^{-1})$. If $g_\tau < 0$ and $g_{\tau\tau} > 0$, the discontinuities are necessarily compressive shock waves (in the case considered, hydraulic jumps with an increasing fluid level), while centered waves involved in solving the Riemann problem (the problem of decay of an initial discontinuity) are necessarily rarefaction waves (reducing the level). If the inequality $g_{\tau\tau} < 0$ holds for certain values of τ, the qualitative behavior of the solutions changes abruptly: along with compression waves, rarefaction shock waves arise, and the solution of the Riemann problem includes compression centered waves. In the case of a nonconvex equation of state, the condition of energy loss (1.10) is insufficient for selecting stable discontinuities, and, therefore, one should introduce additional stability conditions [5].

The equation of state (1.5) satisfies the condition $g_\tau < 0$. The condition $g_{\tau\tau} > 0$ leads to the following inequality for the function $R(p)$:

$$3R(p) - (p - p_0)R'(p) > 0 \qquad (p \geqslant p_0). \tag{2.1}$$

If $R(p) = ap^\alpha$ (the polytropic equation of state, $0 < \alpha < 1$), inequality (2.1) is valid. In the general case, however, the condition $g_{\tau\tau} > 0$ can be violated even in the case of a convex equation of state of a barotropic medium.

Indeed, let the equation of state of a medium have the form

$$\frac{1}{R(p)} = a\xi^{-\alpha} + b(1+\xi)^{-N} \qquad (\xi = p/p_0 > 1),$$

where $a > 0$, $b > 0$, $1 > \alpha > 0$, and $N > 0$. It is obvious that $(1/R(p))' < 0$, $(1/R(p))'' > 0$ and, therefore, this equation of state satisfies the monotonicity and convexity conditions. As $p \to 0$ and $p \to \infty$, the behavior of $R(p)$ is the same as in a polytropic gas. One can easily verify that if for fixed values $\xi > 1$ and $n > 0$ we have

$$N > 3(\xi+1)(\xi-1)^{-1} + n, \quad b > an^{-1}\frac{(1+\xi)^{N+1}}{\xi^\alpha}\left(\frac{3}{\xi-1} - \frac{\alpha}{\xi}\right),$$

then the inequality sign in (2.1) is reversed. Therefore, the "good" properties of the equations of state of the initial barotropic medium do not always ensure similar properties of the equation of state (1.5). Hydraulic jumps in such media can be jumps with a decreasing fluid level.

Let us consider model (1.11). From (1.13) it follows that $[u_\lambda(u - D)] = 0$, $[(u - D)^2] = K$, where K is independent of λ. Let the flow parameters ahead of the jump u_1 and H_1, and the jump-front velocity D satisfy the inequality $u_1 > D$. We determine u_2 and H_2 behind the jump front using the shock relations. According to the previous formula and (1.13), we have

$$u_2 - D = \sqrt{(u_1 - D)^2 - K}, \quad H_2 = \frac{H_1(u_1 - D)}{((u_1 - D)^2 - K)^{1/2}}. \tag{2.2}$$

These formulas are valid for $K \leqslant K_* = \min\limits_\lambda (u_1 - D)^2$. Substitution of (2.2) into the third relation of (1.13) yields the equation

$$F(K) - F(0) = 0 \tag{2.3}$$

$$\left(F(K) = \int\limits_0^1 H_1(u_1 - D)\sqrt{(u_1 - D)^2 - K}\,d\nu + P(\eta_2(K)), \quad \eta_2(K) = \int\limits_0^1 \frac{H_1(u_1 - D)d\nu}{((u_1 - D)^2 - K)^{1/2}}\right).$$

It should be noted that shock relations (1.13) predict that the function $\sigma = [2^{-1}(u - D)^2 + i] = 2^{-1}(u_1 - D)^2 + i_1 - 2^{-1}(u_2 - D)^2 - i_2$ is independent of λ. Therefore this function can be carried out of the integral sign in (1.14), and the condition of energy loss becomes

$$\sigma(K) = i_1 + K/2 - i(\eta_2(K)) > 0. \tag{2.4}$$

The derivative of the function σ is written as

$$\sigma'(K) = \frac{1}{2}\left(1 - \frac{1}{R(\eta_2 + p_0)}\int\limits_0^1 \frac{H_2 d\nu}{(u_2 - D)^2}\right) \tag{2.5}$$

and is related to $F'(K)$ by $F'(K) = -\eta_2(K)\sigma'(K)$. As a consequence of this relation, we obtain the equality

$$\sigma(K) = -\eta_2^{-1}(F(K) - F(0)) - \int\limits_0^K \eta_2'(l)\eta_2^{-2}(l)(F(l) - F(0))dl. \tag{2.6}$$

Let us prove the assertion.

Lemma 1. *Let the function $R(p)$ satisfy inequality* (2.1). *If for some $K_1 < K_*$ the equality $\sigma'(K_1) = 0$ is valid, then necessarily $\sigma''(K_1) < 0$.*

Proof. By the assumption of Lemma,

$$R(\eta_2(K_1) + p_0) = 2\eta_2'(K_1) \tag{2.7}$$

at point K_1. From (2.5) we have

$$\sigma''(K_1) = (2R(\eta_2(K_1) + p_0)^{-1}(R'(\eta_2(K_1) + p_0)\,\eta_2'(K_1) - 2\eta_2''(K_1)).$$

The inequality $\sigma''(K_1) < 0$ is valid, if

$$3\eta_2'^{\,2}(K_1)(\eta_2(K_1))^{-1} - \eta_2''(K_1) < 0 \tag{2.8}$$

[inequality (2.1) and equality (2.7) are used]. But, according to the definition of the function $\eta_2(K)$ [see (2.3)], the relations

$$\eta_2(K) = \int_0^1 \varphi g d\nu, \quad \eta_2'(K) = \frac{1}{2}\int_0^1 \varphi g^3 d\nu,$$

$$\eta_2''(K) = \frac{3}{4}\int_0^1 \varphi g^5 d\nu \quad (\varphi = H_1(u - D), \ \ g = ((u - D)^2 - K)^{-1/2}).$$

are valid. With allowance for the above relations, inequality (2.8) can be reduced to the Cauchy inequality written as

$$\Big(\int_0^1 \varphi g^3 d\nu\Big)^2 \leqslant \int_0^1 \varphi g d\nu \int_0^1 \varphi g^5 d\nu.$$

Lemma is proved.

Corollary 1. *If the equations of state of a barotropic medium satisfy inequality* (2.1), *the function $\sigma(K)$ in the region of $K < K_*$ has only one extremum which is the maximum of $\sigma(K)$.*

In effect, since $\sigma'(K) \to 2^{-1}$ as $K \to -\infty$ and $\sigma'(K) \to -\infty$ as $K \to K_*$ (it is assumed that u_1 and H_1 are smooth functions), the extremum point exists. It follows from the equality $\sigma'(K) = 0$ that $\sigma''(K) < 0$, and, therefore, the extremum point is unique. It is evident that $\sigma(K)$ reaches a maximum value at this point.

Corollary 2. *Let inequality* (2.1) *be valid and K be a root of Eq.* (2.3) that *satisfies the condition $\sigma(K) > 0$. Then necessarily $K > 0$.*

Indeed, in this case $F'(K_1) = 0$ for some $K_1 \in (0, K)$. Because of the relation between the derivatives of the functions σ and F, we conclude that the function $F(K) - F(0)$ reaches its single minimum at point K_1. But, since $F(l) - F(0)$ vanishes when $l = 0$ and $l = K$, this minimum value is negative, and $F(l) - F(0) < 0$ when $l \in (0, K)$. Then, taking into account that (2.6) $\eta_2'(l) > 0$, from equality we find that $K > 0$. In this case, $\sigma'(0) > 0$ and $\sigma'(K) < 0$. Comparison of these inequalities and Eq. (2.5) with the characteristic equation (1.7) shows that the Lax stability conditions (supercriticity of the flow ahead of the jump and subcriticity behind the jump) are satisfied at the front of the discontinuity.

We show that the shock relations enable us to determine the downstream flow parameters, if the upstream flow parameters u_1 and H_1, and the front velocity D are given and satisfy the inequality

$$1 - \frac{1}{R(\eta_1 + p_0)}\int_0^1 H_1(u_1 - D)^{-2} d\nu < 0 \tag{2.9}$$

(the condition of supercriticity of the flow). Suppose that the upstream flow parameters satisfy the additional inequality $F(K_*) - F(0) \geqslant 0$. Then there is a unique root $K_S \in (0, K_*]$ of Eq. (2.3). Indeed, according to (2.9), $F(K) - F(0) < 0$ for small positive K, and, hence, the function $F(K) - F(0)$ vanishes at some point of the interval $(0, K_*]$. The uniqueness of the root and the positivity of $\sigma(K_S)$ are proved above.

Let the upstream flow parameters satisfy the inequality $F(K_*) - F(0) < 0$ at the jump front. In this case Eq. (2.3) does not have roots in the interval $(0, K_*)$ $[F(l) - F(0) < 0$ when $0 < l \leqslant K_*]$, but $\sigma(K_*) > 0$ in accordance with (2.6) and $\sigma'(K_*) < 0$. Let the equality $\lambda = \lambda_*$ $(0 < \lambda_* < 1)$ be valid at the unique point $(u_1(\lambda_*) - D)^2 = K_*$, then $u_2(\lambda_*) - D = 0$. In this case the function

$$H_2 = \frac{H_1(u_1 - D)}{\sqrt{(u_1 - D)^2 - K_*}} + \eta_* \delta(\lambda - \lambda_*) \tag{2.10}$$

$[\delta(\lambda - \lambda_*)$ is a Dirac function, and $\delta(\lambda - \lambda_*)d\lambda$ is the discrete measure concentrated at the point $\lambda = \lambda_*]$ satisfies the shock relations, and the quantity η_2 is defined by

$$\eta_2 = \eta_2(K_*) + \eta_*, \tag{2.11}$$

where η_* is a new unknown quantity, and $\eta_2(K)$ is defined by (2.3). If λ_* coincides with one of the end points of the segment $[0, 1]$, the representation of η_2 as (2.11) is justified by introducing the Stieltjes measure $dp = -Hd\lambda$, which is generated by the jump function and an absolutely continuous function. In this case, an analog of relation (2.10) can be obtained for the measures dp_1 and dp_2.

The conservation law for the total momentum of the fluid layer and the total energy loss condition are of the form

$$P(\eta_2(K_*) + \eta_*) - P(\eta_2(K_*)) = -F(K_*) + F(0), \quad i(\eta_2(K_*) + \eta_*) - i(\eta_2(K_*)) \leqslant \sigma(K_*). \tag{2.12}$$

The first equation determines uniquely the value $\eta_* > 0$ $[P(\eta)$ is the monotone function, $P(\eta) \to \infty$ as $\eta \to \infty]$. We show that for the resulting η_* the second inequality (2.12) is valid. According to (2.6), we have

$$\sigma(K_*) \geqslant (\eta_2(K_*))^{-1}(F(0) - F(K_*)),$$

and, hence, it is sufficient to prove that

$$i(\eta_2(K_*) - i(\eta_2(K_*)) \leqslant (\eta_2(K_*))^{-1}(P(\eta_2(K_*) + \eta) - P(\eta_2(K_*)).$$

Representation of the increments of the functions by the integrals of their derivatives leads to the obvious inequality

$$\int_{\eta_2}^{\eta_2+\eta_*} \frac{1}{\eta} p'(\eta) d\eta \leqslant \frac{1}{\eta_2} \int_{\eta_2}^{\eta_2+\eta_*} p'(\eta) d\eta.$$

Consequently, the condition of energy loss is satisfied. As a result, when $F(K_*) - F(0) < 0$, the downstream velocity is given by (2.2) (in which one should set $K = K_*$), and H_2 is given by (2.10). The bottom pressure coincides with $\eta_2 + p_0$, where η_2 is determined in (2.11). When $Y = 0$, the first relation of (1.3) determines the depth of the fluid layer h. The equations

$$f(h - Y_1) = \int_{\lambda_*}^{1} \frac{H_1(u_1 - D)d\nu}{\sqrt{(u_1 - D)^2 - K_*}} + \eta_*; \tag{2.13a}$$

$$f(h - Y_2) = \int_{\lambda_*}^{1} \frac{H_1(u_1 - D)d\nu}{\sqrt{(u_1 - D)^2 - K_*}} \tag{2.13b}$$

give the depth Y_1 and Y_2, where $u_2 - D$ vanishes. In the layer $Y_1 \leqslant Y \leqslant Y_2$, the equality $u_2 - D = 0$ is valid, and the density is given by the relation $\rho = f'(h - Y)$. If we substitute λ for λ_* in (2.13), then (2.13a) determines the function $Y_1(\lambda)$ for $\lambda < \lambda_*$, and relation (2.13b) determines the function $Y_2(\lambda)$. Equalities (2.2) and $Y = Y_1(\lambda)$ $(0 \leqslant \lambda \leqslant \lambda_*)$ specify the function $u_2(Y)$ at the discontinuity front when $0 \leqslant Y \leqslant Y_1$. Similarly, formula (2.2) and $Y = Y_2(\lambda)$ define $u_2(Y)$ for $Y_2 \leqslant Y \leqslant h$. If the upstream flow is characterized by a monotonic velocity profile, then either $Y_1 = 0$ or $Y_2 = h$. The resulting solution describes the flow with a stagnant (with respect to the jump front) zone $Y_1 \leqslant Y \leqslant Y_2$. The occurrence of this zone is caused by the formation of either the recirculating-flow zone near the bottom ($Y_1 = 0$) or a "roller" in the vicinity of free boundary ($Y_2 = h$).

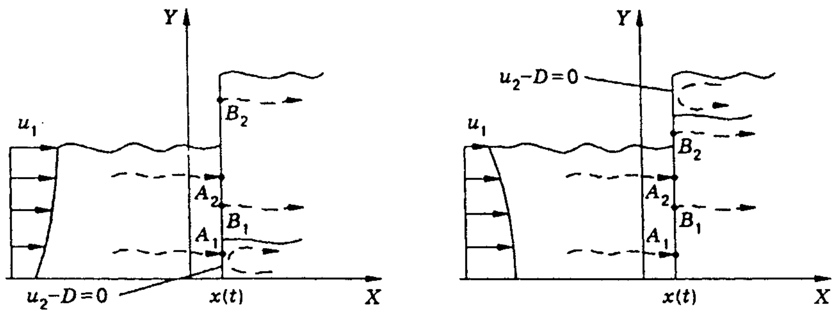

Fig. 1

The possible flow patterns for a monotonic velocity profile are presented in Fig. 1. If $F(K_*) - F(0) \geqslant 0$, there are no stagnation zones, and the fluid layer behind the front consists of the particles flowing through the jump front. If the minimum value of $(u_1 - D)^2 = K_*$ is reached at several points λ, one can construct a solution with several stagnation zones. In this case, the momentum balance equation permits one to find only the sum of the pressure gradients η_*. The flow in the interlayers and their thicknesses are generally determined by the conditions to the right of the discontinuity front. It should be noted that for a barotropic medium with a convex [in the sense of inequality (2.1)] equation of state the qualitative properties of a hydraulic jump are similar to those for an incompressible fluid [1].

In the general case, when inequality (2.1) can be violated, we introduce the concept of an admissible discontinuity to leave structurally unstable discontinuities out of consideration [5]. A discontinuity is called admissible if for the corresponding root K of Eq. (2.3) the condition

$$K(F(l) - F(0)) \leqslant 0 \qquad [l \in (0, K)] \tag{2.14}$$

is satisfied. This condition implies that the graph of the function $Z(l) = F(l) - F(0)$ with $l \in (0, K)$ lies in the half-plane $Z \leqslant 0$ if $K > 0$, and in the half-plane $Z \geqslant 0$ if $K < 0$. The condition of energy loss $\sigma(K) > 0$ and the Lax stability conditions $\sigma'(0) \geqslant 0$ and $\sigma'(K) \leqslant 0$ in weakened form (the equality is allowed) follow from (2.14).

Let us assume that the upstream-flow parameters at the jump front satisfy either the condition $\sigma'(0) > 0$ or the conditions $\sigma'(0) = 0$ and $\sigma''(0) > 0$. When the inequality $F(K_*) - F(0) \geqslant 0$ holds, one can show that a value $K > 0$ exists that determines the admissible jump raising the fluid level without stagnation zones at the front. If $F(K_*) - F(0) < 0$, the solution of the discontinuity relations exists as well, but it can determine the jump with stagnation zones behind the front [in the latter case inequality (2.14) should hold when $K = K_*$]. The jumps decreasing the flow level can appear in certain regions of the upstream flow parameters: if $(F(l) - F(0)) \leqslant 0$ for a certain $l < 0$, we can find $K < 0$ and the downstream-flow parameters at the front of an admissible jump with a decreasing fluid level. These facts are proved by the same arguments as presented above. Justification of the admissibility condition (2.14) should be considered separately. It should be noted that the proposed condition is an analog of the admissibility condition for gas-dynamic discontinuities [5].

Thus it is established that for a barotropic medium with an equation of state that satisfies (2.1) the jump relations (1.13) and the energy-loss condition (1.14) make it possible to determine the flow parameters behind a hydraulic jump propagating at a given supercritical velocity relative to the incident flow. If the upstream flow velocity profile is monotonic ($u_Y \neq 0$), the parameters are determined uniquely. For media foe which the equations of state do not satisfy condition (2.1), the discontinuity relations (1.13) and the admissibility condition (2.14) also allows one to determine (strictly speaking, uniquely) the downstream state at the discontinuity front moving with, at least, critical velocity. Along with the jumps increasing the fluid level, the jumps lowering the level can arise. This implies that in modeling large-scale processes (for example, atmospheric) the properties of the equations of state for the medium influence significantly the flow pattern.

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